

DISTRIBUTED STOCHASTIC MODEL PREDICTIVE CONTROL SYNTHESIS FOR LARGE-SCALE UNCERTAIN LINEAR SYSTEMS

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ABSTRACT. This paper presents an approach to distributed stochastic model predictive control (SMPC) of large-scale linear systems with additive disturbances and multiplicative uncertainties in a plug-and-play (PnP) framework. Typical SMPC approaches for such problems involve formulating a large-scale finite-horizon chance-constrained optimization problem at each sampling time, which is in general non-convex and difficult to solve. Using an approximation, the so-called scenario approach, we formulate a large-scale scenario program and provide a theoretical guarantee to quantify the robustness of the obtained solution. However, such a reformulation leads to a computational tractability issue, due to the large number of required scenarios. To this end, we present two novel ideas in this paper to address this issue. We first provide a technique to decompose the large-scale scenario program into distributed scenario programs that exchange a certain number of scenarios with each other in order to compute local decisions. We show the exactness of the decomposition with a-priori probabilistic guarantees for the desired level of constraint fulfillment. As our second contribution, we develop an inter-agent soft communication scheme based on a set parametrization technique together with the notion of probabilistically reliable set to reduce the required communication between each subproblem. We show how to incorporate the probabilistic reliability notion into existing results and provide new guarantees for the desired level of constraint violations. A simulation study is presented to illustrate the advantages of our proposed framework.

Keywords. Distributed MPC, Stochastic MPC, Scenario MPC, Plug-and-Play Framework.

1. INTRODUCTION

Stochastic model predictive control (SMPC) has attracted significant attention in the control literature, due to its functionality to differently handle uncertain systems. SMPC takes into account the stochastic characteristics of the uncertainties and thereby the system constraints are treated in a probabilistic sense, i.e. using chance constraints [1, 2]. SMPC computes an optimal control sequence that minimizes a given objective function subject to the uncertain system dynamics model and chance constraints in a receding horizon fashion [3]. Chance constraints enable SMPC to offer an alternative approach to achieve a less conservative solution compared to robust model predictive control (MPC) [4], since it directly incorporates the tradeoff between constraint feasibility and control performance.

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Distributed MPC has been an active research area in the past decades, due to its applicability in different domains such as power networks [5], chemical plants [6], process control [7], and building automation [8]. For such large-scale dynamic systems with state and input constraints, distributed MPC is an attractive control scheme. In distributed MPC one replaces large-scale optimization problems stemming from centralized MPC with several smaller-scale problems that can be solved in parallel. These problems make use of pieces of information from other subsystems to design a distributed MPC. In the presence of uncertainties, however, the main challenge in the formulation of distributed MPC is how the controllers should exchange pieces of information through a communication scheme among subsystems (see, e.g. [9], and references therein). This highlights the necessity of developing distributed control strategies to cope with the uncertainties in subsystems while at the same time minimizing information exchange through a communication framework.

To handle uncertainties in distributed MPC, some approaches are based on robust MPC [10, 11]. Assuming that the uncertainty is bounded, a robust optimization problem is solved at each sampling time, leading to a control law that satisfies the constraints for all admissible values of the uncertainty. The resulting solution using such an approach tends to be very conservative in many cases. Tube-based MPC, see for example [12] and the references therein, was considered in a plug-and-play (PnP) decentralized setup in [13], and it has been recently extended to distributed control systems [14] for a collection of linear stochastic subsystems with independent dynamics. While in [14] coupled chance constraints were considered separately at each sampling time, in this paper we consider a chance constraint on the feasibility of trajectories of dynamically coupled subsystems. Our approach is motivated by [13] to reduce the conservativeness of the control design. Other representative approaches for SMPC of a single stochastic system include affine parametrization of the control policy [15], the randomized (scenario) approach [16–19], and the combined randomized and robust approach [20–22]. None of these approaches, to the best of our knowledge, have been considered in distributed control strategies.

This paper aims to establish some fundamental techniques for distributed SMPC by promoting the scenario MPC technique to the distributed case in a more systematical approach. Scenario MPC approximates SMPC via the so-called scenario (sample) approach [23, 24], and if the underlying optimization problem is convex with respect to the decision variables, finite sample guarantees can be provided. Following such an approach, the computation time for a realistic large-scale system of interest becomes prohibitive, due to the fact that the number of samples to be extracted tends to be very high, and consequently leads to a large number of constraints in the resulting optimization problem. To overcome the computational burden caused by the large number of constraints in [25, 26] a heuristic sampled-based approach was used in an iterative distributed fashion via dual decomposition such that all subsystems collaboratively optimize a global performance index. In another interesting recent work [27], a distributed multi-agent consensus algorithm was presented to achieve consensus on a common value of the decision vector subject to random constraints. The authors established a probabilistic bound on the tails of the consensus violation, similarly to [28]. In addition, they studied a case where the constraints of agents are affected by a common uncertainty source and derived a-posteriori probabilistic bounds on the number of shared scenarios based on the results in [29] to reduce significantly the conservativeness of the resulting solution. We however note

that in most of the aforementioned references the aim to reduce communication among subsystems, which we refer to as agents, has not been addressed.

Our work in this paper differs from the aforesaid references in two important aspects. A decomposition technique based on the large-scale system dynamics is employed to distribute the resulting centralized scenario optimization problem at each sampling time and a novel communication scheme is introduced to reduce the communication between the small-scale problems. The main contributions of this paper are twofold:

- a) We provide a technique to decompose the large-scale scenario program into distributed scenario programs that exchange a certain number of scenarios with each other in order to compute local decisions. We show the exactness of the decomposition with a-priori probabilistic guarantees for the desired level of constraint fulfillment.
- b) We develop an inter-agent soft communication scheme based on a set parametrization technique together with the notion of probabilistically reliable set to reduce the required communication between each subproblem. We show how to incorporate the probabilistic reliability notion into existing results and provide new guarantees for the desired level of constraint violations.

The structure of this paper is as follows: Section 2 describes a mathematical model of the control system dynamics together with some preparatory results. In this section, we first formulate a SMPC for a large-scale uncertain linear system with additive disturbance, then provide a reformulation, namely the centralized scenario MPC, and provide a theoretical study on the connections of these two control problems. In Section 3, we propose a decomposition technique to distributed scenario MPC, and analyze the robustness of the obtained solution compared to the centralized scenario MPC formulated in the previous section. Section 4 introduces two types of inter-agent communication schemes between each subproblem, namely hard and soft communications. We then quantify the robustness of the proposed schemes and provide a new theoretical guarantee. Section 5 provides a summary on our developments to establish an applied PnP distributed SMPC framework, and Section 6 presents a simulation study to illustrate the functionality of our theoretical achievements. In Section 7 we conclude this paper with some remarks and future work.

NOTATIONS

\mathbb{R}, \mathbb{R}_+ denote the real and positive real numbers, and \mathbb{N}, \mathbb{N}_+ the natural and positive natural numbers, respectively. We operate within n -dimensional space \mathbb{R}^n composed by column vectors $u, v \in \mathbb{R}^n$. The Cartesian product over n sets $\mathcal{X}_1, \dots, \mathcal{X}_n$ is given by: $\prod_{i=1}^n \mathcal{X}_i = \mathcal{X}_1 \times \dots \times \mathcal{X}_n = \{(x_1, \dots, x_n) : x_i \in \mathcal{X}_i\}$. The cardinality of a set \mathcal{A} is shown by $|\mathcal{A}| = A$. We denote a block-diagonal matrix with blocks X_i , $i \in \{1, \dots, n\}$, by $\text{diag}_{i \in \{1, \dots, n\}}(X_i)$, and a vector consisting of stacked sub vectors x_i , $i \in \{1, \dots, n\}$, by $\text{col}_{i \in \{1, \dots, n\}}(x_i)$.

Given a metric space Δ , its Borel σ -algebra is denoted by $\mathfrak{B}(\Delta)$. Throughout the paper, measurability always refers to Borel measurability. In a probability space $(\Delta, \mathfrak{B}(\Delta), \mathbb{P})$, we denote the N -Cartesian product set of Δ by Δ^N with the respective product measure by \mathbb{P}^N .

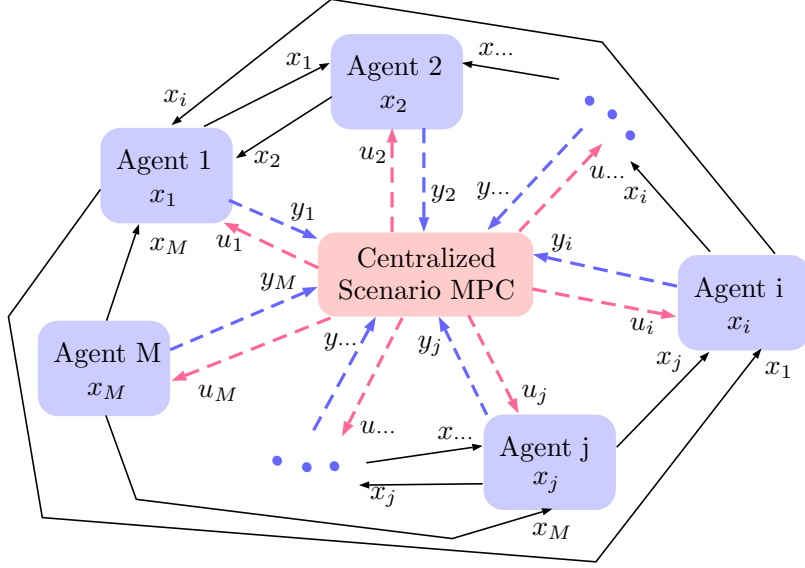


FIGURE 1. Centralized scenario MPC that corresponds to the problem (7).

2. PROBLEM STATEMENT AND PREPARATORY RESULTS

This section provides an overview on the control problem statement. We first describe a large-scale uncertain linear system dynamic together with input and state constraint sets and the control objective. We then formulate a centralized SMPC for such a large-scale control system problem. Finally a tractable reformulation based on the scenario MPC [30] together with theoretical connections are provided.

Consider a discrete-time uncertain linear system with additive disturbance in a compact form as follows:

$$x_{k+1} = A(\delta_k)x_k + B(\delta_k)u_k + C(\delta_k)w_k, \quad (1)$$

with a fixed initial condition $x_0 \in \mathbb{R}^m$. Here $k \in \mathcal{T} := \{0, 1, \dots, T-1\}$ denotes instance of time, $x_k \in \mathcal{X} \subset \mathbb{R}^m$ and $u_k \in \mathcal{U} \subset \mathbb{R}^p$ correspond to the state and control input, respectively, and $w_k \in \mathbb{R}^n$ is related to an additive disturbance. The system parameters $A(\delta_k) \in \mathbb{R}^{m \times m}$ and the control input $B(\delta_k) \in \mathbb{R}^{m \times p}$ as well as $C(\delta_k) \in \mathbb{R}^{m \times n}$ are random, since they are known functions of an uncertain variable δ_k that influences the system parameters at each time step k .

Assumption 1. $\mathbf{w} := \{w_k\}_{k \in \mathcal{T}}$ and $\boldsymbol{\delta} := \{\delta_k\}_{k \in \mathcal{T}}$ are defined on probability spaces $(\mathcal{W}, \mathfrak{B}(\mathcal{W}), \mathbb{P}_{\mathbf{w}})$ and $(\Delta, \mathfrak{B}(\Delta), \mathbb{P}_{\boldsymbol{\delta}})$, respectively. \mathbf{w} and $\boldsymbol{\delta}$ are two independent random processes, where $\mathbb{P}_{\mathbf{w}}$ and $\mathbb{P}_{\boldsymbol{\delta}}$ are two different probability measures defined over $\mathcal{W} \subseteq \mathbb{R}^q$ and $\Delta \subseteq \mathbb{R}^m$, respectively, and $\mathfrak{B}(\cdot)$ denotes a Borel σ -algebra.

The support sets \mathcal{W} and Δ of \mathbf{w} and $\boldsymbol{\delta}$, respectively, together with their probability measures $\mathbb{P}_{\mathbf{w}}$ and $\mathbb{P}_{\boldsymbol{\delta}}$ are entirely generic. In fact, \mathcal{W} , Δ and $\mathbb{P}_{\mathbf{w}}$, $\mathbb{P}_{\boldsymbol{\delta}}$ do not need to be known explicitly. Instead, the only requirement is availability of a "sufficient number" of samples, and will become concrete in later parts of the paper.

The system in (1) is subject to hard constraints on the system state trajectories and control input. Consider the state and control input constraint sets of the form

$$\mathcal{X} := \{x \in \mathbb{R}^m : Gx \leq g\}, \mathcal{U} := \{u \in \mathbb{R}^p : Hu \leq h\},$$

where $G \in \mathbb{R}^{q \times m}$, $g \in \mathbb{R}^q$, and $H \in \mathbb{R}^{r \times p}$, $h \in \mathbb{R}^r$. Keeping the state inside a feasible set $\mathcal{X} \subset \mathbb{R}^m$ for the entire prediction horizon steps maybe too conservative and results in a poor performance. In particular, this is the case when the best performance is achieved close to the boundary of \mathcal{X} , and thus, constraint violations will be unavoidable due to the fact that the system parameters are imperfect and uncertain (1). To tackle such a problem, we will consider chance constraints on the state trajectories to avoid violation of the state variables constraints probabilistically even if the disturbance \mathbf{w} or uncertainty δ has unbounded support. Notice that a robust problem formulation [4] cannot cope with problems having an unbounded disturbance set.

The control task is to choose the control inputs $\mathbf{u} := \{u_k\}_{k \in \mathcal{T}}$ from an input constraint set \mathcal{U} via a state feedback law $\phi : \mathbb{R}^m \rightarrow \mathcal{U}$ with $u_k = \phi(x_k)$. In order to find such a stabilizing full-information controller that leads to admissible control inputs and satisfies the state constraints, we follow the traditional MPC approach. The design relies on the standard assumption of the existence of a suitable pre-stabilizing control law, e.g., [13, Proposition 1]. To cope with the state prediction under uncertainty and disturbance, we employ a parametrized feedback policy [15] for (1) and split the control input, $u_k = Kx_k + v_k$, with v_k as a free correction input variable to compensate for disturbances.

The control objective is to determine a receding horizon control, which minimizes a cumulative quadratic stage cost of an infinite horizon that is as follows:

$$J_\infty := \lim_{t \rightarrow \infty} \frac{1}{t} \sum_{k=0}^t \mathbb{E} \left[\mathbf{x}_k^\top Q \mathbf{x}_k + \mathbf{u}_k^\top R \mathbf{u}_k \right], \quad (2)$$

where $\mathbf{x} := \{x_k\}_{k \in \mathcal{T}}$, $Q \in \mathbb{R}_{\geq 0}^{m \times m}$, $R \in \mathbb{R}_{> 0}^{p \times p}$, and it is assumed that $(A, Q^{\frac{1}{2}})$ is detectable. Each stage cost term is taken in expectation $\mathbb{E}[\cdot]$, since the argument x_k is a random variable. The formulated cost function in (2) has a very large time horizon $t \in \mathcal{T}$, i.e. the life cycle of the system [18], and particularly, it is much larger than any practical prediction horizon $T \in \mathbb{N}$ ($t \gg T$). Therefore, the control problem cannot be solved for the entire prediction horizon $t \in \mathcal{T}$. We therefore recast J_∞ into a finite horizon cost $J(\cdot) : \mathbb{R}^m \times \mathbb{R}^p \rightarrow \mathbb{R}$ to be minimized at time k and is defined as

$$J(\mathbf{x}, \mathbf{u}) := \mathbb{E} \left[\sum_{k=0}^{T-1} \left(\mathbf{x}_k^\top Q \mathbf{x}_k + \mathbf{u}_k^\top R \mathbf{u}_k \right) + \mathbf{x}_T^\top P \mathbf{x}_T \right], \quad (3)$$

and P is the solution of discrete-time Lyapunov equation:

$$\mathbb{E}[A_{cl}(\delta_k)^\top P A_{cl}(\delta_k)] + Q + K^\top R K - P \preceq 0, \quad (4)$$

for the closed-loop system, e.g., $A_{cl}(\delta_k) = A(\delta_k) + B(\delta_k)K$.

Using $\mathbf{v} = \{v_k\}_{k \in \mathcal{T}}$, consider now the following stochastic control problem as follows:

$$\min_{\mathbf{v} \in \mathbb{R}^{Tm}} J(\mathbf{x}, \mathbf{u}) \quad (5a)$$

$$\text{s.t. } x_{k+1} = A(\delta_k)x_k + B(\delta_k)u_k + C(\delta_k)w_k, \quad x_k = x_0, \quad (5b)$$

$$\mathbb{P}[x_{k+\ell} \in \mathcal{X}, \ell \in \mathbb{N}_+] \geq 1 - \varepsilon, \quad (5c)$$

$$u_k = Kx_k + v_k \in \mathcal{U}, \quad \forall k \in \mathcal{T}, \quad (5d)$$

where $\varepsilon \in (0, 1)$ is the admissible state constraint violation parameter of the large-scale system (1).

Remark 1. The objective function is assumed to be a quadratic function, however, this is not a restriction and any generic convex function can be chosen instead. It is important to mention that the parameters of constraint sets, \mathcal{X} , \mathcal{U} , and the objective function $J(\cdot)$ can be time-varying with respect to each sampling time $k \in \mathcal{T}$. For the clarity of our problem formulation, we assume time-invariance.

Remark 2. The state trajectory $x_{k+\ell}$, $\forall \ell \in \mathbb{N}_+$, has a dependency on the random variables \mathbf{w} and δ , and thus, the chance constraint can be interpreted as follows: the probability of violating the state constraint at the future time step $\ell \in \mathbb{N}_+$ is restricted to ε , given that the state of the system in (1) is measurable at each time step $k \in \mathcal{T}$. Even though \mathcal{U} and \mathcal{X} are convex compact sets, due to the chance constraint on the state trajectory, the feasible set of the optimization problem in (5) is a non-convex set, in general.

Remark 3. Instead of the chance constraint on the state trajectory of form (5c), one can also bound the average rate of state constraint violations [30]. Moreover, one can also define the cost function (5a) as a desired quantile of the sum of discounted stage costs ("value-at-risk"), instead of the sum of expected values.

Remark 4. Instead of a state feedback law, one can also consider a nonlinear disturbance parametrization feedback policy over the prediction horizon, similar to [22], using the scenario-based approximation. Such a parametrization does not affect the convexity of the resulting optimization problem [16].

To handle the chance constraint (5c), we recall a scenario-based approximation [30]. w_k and δ_k at each sampling time $k \in \mathcal{T}$ are not necessarily independent and identically distributed (i.i.d.). Particularly, they may have time-varying distributions and/or be correlated in time. We assume that a "sufficient number" of i.i.d. samples of the string of disturbance $\mathbf{w} \in \mathcal{W}$ and $\delta \in \Delta$ can be obtained either empirically or by a random number generator. We denote $\mathcal{S}_{\mathbf{w}} := \{\mathbf{w}^{(1)}, \dots, \mathbf{w}^{(N_s)}\} \in \mathcal{W}^{N_s}$ and $\mathcal{S}_{\delta} := \{\delta^{(1)}, \dots, \delta^{(N_s)}\} \in \Delta^{N_s}$ as a set of given finite samples (scenarios).

Following the approach in [6], we approximate the expected value of the objective function empirically by averaging the value of its argument for some number of different scenarios, which plays a tuning parameter role. Using $N_{\bar{s}}$ as the tuning parameter, consider $N_{\bar{s}}$ number of different scenarios of \mathbf{w} and δ to build $\bar{\mathcal{S}}_{\mathbf{w}} = \{\mathbf{w}^{(1)}, \dots, \mathbf{w}^{(N_{\bar{s}})}\} \subset \mathcal{W}^{N_{\bar{s}}}$ and $\bar{\mathcal{S}}_{\delta} := \{\delta^{(1)}, \dots, \delta^{(N_{\bar{s}})}\} \in \Delta^{N_{\bar{s}}}$, respectively. We then approximate the cost function empirically as follows:

$$J(\mathbf{x}, \mathbf{u}) = \mathbb{E}_{\{\mathbf{w}, \delta\} \in \mathcal{W} \times \Delta} \left[\sum_{k=0}^{T-1} V(x_k(w_k, \delta_k), u_k) \right] \approx \sum_{\{\mathbf{w}^{(i)}, \delta^{(i)}\} \in \bar{\mathcal{S}}_{\mathbf{w}} \times \bar{\mathcal{S}}_{\delta}} \sum_{k=0}^{T-1} V(x_k(w_k^{(i)}, \delta_k^{(i)}), u_k), \quad (6)$$

where $V(x_k(w, \delta_k), u_k) = (x_k(w_k, \delta_k))^\top Q x_k(w_k, \delta_k) + u_k^\top R u_k + x_T(w_k, \delta_k)^\top P x_T(w_k, \delta_k)$. Note that $x_k(w_k, \delta_k)$ denotes the dependency of the state variables on the random variables.

We are now in a position to formulate a tractable version of the proposed stochastic control problem in (5) using the following finite horizon scenario program:

$$\min_{\mathbf{v} \in \mathbb{R}^{Tm}} J(\mathbf{x}, \mathbf{u}) \quad (7a)$$

$$\text{s.t. } x_{k+1}^{(i)} = A(\delta_k^{(i)})x_k^{(i)} + B(\delta_k^{(i)})u_k^{(i)} + C(\delta_k^{(i)})w_k^{(i)}, \quad x_k^{(i)} = x_0, \quad (7b)$$

$$x_{k+\ell}^{(i)} \in \mathcal{X}, \ell \in \mathbb{N}_+, \forall \mathbf{w}^{(i)} \in \mathcal{S}_{\mathbf{w}}, \forall \delta^{(i)} \in \mathcal{S}_{\delta} \quad (7c)$$

$$u_k^{(i)} = K x_k^{(i)} + v_k \in \mathcal{U}, \quad \forall k \in \mathcal{T}. \quad (7d)$$

The solution of (7) is the optimal control input sequence $\mathbf{v}^* = \{v_k^*, \dots, v_{k+T-1}^*\}$. Based on an MPC paradigm, the current input is implemented as $u_k := K x_k + v_k^*$ and we proceed in a receding horizon fashion. This means (7) is solved at each time step k by using the current measurement of the state x_k . Note that new scenarios are needed to be generated at each sampling time $k \in \mathcal{T}$.

The key features of the proposed tractable optimization problem (7) are as follows:

- there is no need to know the probability measures $\mathbb{P}_{\mathbf{w}}$ and \mathbb{P}_{δ} explicitly, only the capability of obtaining random scenarios is enough.
- formal results to quantify the robustness of the obtained approximations are available. In particular the results follow the so-called scenario approach [24], which allow to bound a-priori the violation probability of the obtained solution via (7).

In the following theorem, we restate the explicit theoretical bound of [24, Theorem 1] which measures the finite scenarios behavior of (7).

Theorem 1. Let $\beta \in (0, 1)$ and $N_s \geq N(\varepsilon, \beta, Tm)$, where

$$N(\varepsilon, \beta, Tm) := \min \left\{ N \in \mathbb{N} \mid \sum_{i=0}^{Tm-1} \binom{N}{i} \varepsilon^i (1 - \varepsilon)^{N-i} \leq \beta \right\}.$$

If the optimizer of the problem (7), \mathbf{v}^* , is applied to the discrete time dynamical system (1) for a finite horizon of length T , then, with at least confidence $1 - \beta$, the original chance constraint (5c) is satisfied for all $k \in \mathcal{T}$.

It was shown in [24] that the above bound is tight. The interpretation of Theorem 1 is as follows: when applying \mathbf{v}^* in a finite horizon control problem, the violation of the feasibility of the state trajectory remains below ε with confidence $1 - \beta$:

$$\mathbb{P}^{N_s} [\mathcal{S}_{\mathbf{w}} \in \mathcal{W}^{N_s}, \mathcal{S}_{\delta} \in \Delta^{N_s} : \text{Vio}(\mathbf{v}^*) \leq \varepsilon] \geq 1 - \beta,$$

with

$$\text{Vio}(\mathbf{v}^*) := \mathbb{P} [\mathbf{w} \in \mathcal{W}, \delta \in \Delta : x_{k+\ell} = A_{cl}(\delta_k)x_k + B(\delta_k)v_k^* + C(\delta_k)w_k \notin \mathcal{X}, \ell \in \mathbb{N}_+ \mid x_k = x_0],$$

where $A_{cl}(\delta_k) = A(\delta_k) + B(\delta_k)K$. It is worth to mention that the proposed hard constraint on the control input in (7d) is also met in a probabilistic sense, due to the nature of the scenario approach that appears in the proposed optimization problem (7).

Remark 5. One can obtain an explicit expression for the desired number of scenarios N_s as in [31], where it is shown that given $\varepsilon, \beta \in (0, 1)$ then $N_s \geq \frac{e}{e-1} \frac{1}{\varepsilon} \left(Tm + \ln \frac{1}{\beta} \right)$. It is important to mention that N_s is used to construct the sets of scenarios, $\mathcal{S}_w, \mathcal{S}_\delta$ to obtain a probabilistic guarantee for the desired level of feasibility. The number of scenarios N_s can be interpreted as a tuning variable to approximate the objective function empirically.

We formulated a large-scale SMPC (5c) together with a tractable reformulation based on the proposed centralized scenario MPC (7). Figure 1 shows a pictorial representation of (7) as a large-scale network of interconnected agents to summarize this section. In the following section, we will provide a distributed framework to solve the proposed problem in (7) by decomposing the large-scale system dynamics (1).

3. DISTRIBUTED SCENARIO MODEL PREDICTIVE CONTROL

In this section, we first describe a decomposition technique to partition the large-scale system dynamics in (1) that we will get benefit in the subsequent sections. We then present the theoretical links to the results that we provided in the previous section.

Consider the system (1) that is assumed to be decomposable into M subsystems and let $\mathcal{M} = \{1, 2, \dots, M\}$ be the set of subsystem indices (for more details on the decomposition techniques of a large-scale system, we refer interested reader to [32, Chapter 3]). The state variables x_k , control input signals u_k and the additive disturbance w_k , can be considered as $x_k = \text{col}_{i \in \mathcal{M}}(x_{i,k})$, $u_k = \text{col}_{i \in \mathcal{M}}(u_{i,k})$, and $w_k = \text{col}_{i \in \mathcal{M}}(w_{i,k})$, respectively, where $x_{i,k} \in \mathbb{R}^{m_i}$, $u_{i,k} \in \mathbb{R}^{p_i}$, $w_{i,k} \in \mathbb{R}^{n_i}$, and $\sum_{i \in \mathcal{M}} m_i = m$, $\sum_{i \in \mathcal{M}} p_i = p$, $\sum_{i \in \mathcal{M}} n_i = n$. The following assumption is important in order to be able to partition the system parameters.

Assumption 2. It is assumed that the control input and the disturbance variables of the subsystems are decoupled, e.g. $u_{i,k}$ and $w_{i,k}$ only affect subsystem $i \in \mathcal{M}$ for all $k \in \mathcal{T}$.

We are now able to decompose the large-scale system matrices $A(\delta_k) \in \mathbb{R}^{m \times m}$, $B(\delta_k) \in \mathbb{R}^{m \times p}$, as well as $C(\delta_k) \in \mathbb{R}^{m \times n}$ as follows:

$$A(\delta_k) = \begin{bmatrix} A_{11}(\delta_k) & \cdots & A_{1m}(\delta_k) \\ \vdots & \ddots & \vdots \\ A_{m1}(\delta_k) & \cdots & A_{mm}(\delta_k) \end{bmatrix}, \quad B(\delta_k) = \begin{bmatrix} B_{11}(\delta_k) & \cdots & \mathbf{0} \\ \vdots & \ddots & \vdots \\ \mathbf{0} & \cdots & B_{mm}(\delta_k) \end{bmatrix}, \quad C(\delta_k) = \begin{bmatrix} C_1(\delta_k) \\ \vdots \\ C_m(\delta_k) \end{bmatrix},$$

where $A_{ij}(\delta_k) \in \mathbb{R}^{m_i \times m_j}$, $B_{ii}(\delta_k) \in \mathbb{R}^{m_i \times p_i}$, and $C_i(\delta_k) \in \mathbb{R}^{m_i \times n_i}$. Define the set of neighboring subsystems of subsystem i as follows:

$$\mathcal{N}_i = \{j \in \mathcal{M} \setminus i \mid A_{ij}(\delta_k) \neq \mathbf{0}\}, \quad (8)$$

where $\mathbf{0}$ denotes a zero matrix with proper dimension. Consider now a large-scale network that consists of M interconnected subsystems, and each subsystem can be described by a linear uncertain discrete-time invariant system with additive disturbance of the form

$$\begin{cases} x_{i,k+1} &= A_{ii}(\delta_k)x_{i,k} + B_{ii}(\delta_k)u_{i,k} + q_{i,k} \\ q_{i,k} &= \sum_{j \in \mathcal{N}_i} A_{ij}(\delta_k)x_{j,k} + C_i(\delta_k)w_{i,k} \end{cases}, \quad (9)$$

where for each subsystem i , $x_{i,k} \in \mathcal{X}_i \subseteq \mathbb{R}^{m_i}$, $u_{i,k} \in \mathcal{U}_i \subseteq \mathbb{R}^{p_i}$, and $w_{i,k} \in \mathbb{R}^{n_i}$. We define the state and control input constraint sets of each subsystem $i \in \mathcal{M}$ as the following form:

$$\mathcal{X}_i := \{x \in \mathbb{R}^{m_i} : [G]_i x \leq [g]_i\}, \quad \mathcal{U}_i := \{u \in \mathbb{R}^{p_i} : [H]_i u \leq [h]_i\},$$

such that $\mathcal{X} = \prod_{i \in \mathcal{M}} \mathcal{X}_i$, and $\mathcal{U} = \prod_{i \in \mathcal{M}} \mathcal{U}_i$. Following Assumption 2, one can consider a linear feedback gain matrix K_i for each subsystem $i \in \mathcal{M}$ such that $K = \text{diag}_{i \in \mathcal{M}}(K_i)$. Using K_i in each subsystem, we assume that there exists P_i for each subsystem $i \in \mathcal{M}$ such that $P = \text{diag}_{i \in \mathcal{M}}(P_i)$ to preserve the condition in (4). Consider now the objective function of each subsystem $i \in \mathcal{M}$ in the following form:

$$J_i(\mathbf{x}_i, \mathbf{u}_i) := \mathbb{E} \left[\sum_{k=0}^{T-1} \left(x_{i,k}^\top Q_i x_{i,k} + u_{i,k}^\top R_i u_{i,k} \right) + x_{i,T}^\top P_i x_{i,T} \right],$$

where $Q_i \in \mathbb{R}_{\geq 0}^{m_i \times m_i}$, $R_i \in \mathbb{R}_{> 0}^{p_i \times p_i}$ such that $Q = \text{diag}_{i \in \mathcal{M}}(Q_i)$, and $R = \text{diag}_{i \in \mathcal{M}}(R_i)$. Note that $\mathbf{x}_i = \text{col}_{k \in \mathcal{T}}(x_{i,k})$ and $\mathbf{u}_i = \text{col}_{k \in \mathcal{T}}(u_{i,k})$ such that $\mathbf{x} = \text{col}_{i \in \mathcal{M}}(\mathbf{x}_i)$ and $\mathbf{u} = \text{col}_{i \in \mathcal{M}}(\mathbf{u}_i)$.

Remark 6. For sake of simplicity of the mathematical notations, a decomposition of multiplicative uncertainty δ_k is not considered. We however note that such a decomposition is straightforward by considering $\delta_{i,k}$ for each subsystem $i \in \mathcal{M}$. This leads to $A_{ii}(\delta_{i,k})$, $B_{ii}(\delta_{i,k})$, $C_i(\delta_{i,k})$, and an effect on the state coupling matrices between subsystems $A_{ij}(\delta_{i,k})$ for all $j \in \mathcal{N}_i$ in each subsystem $i \in \mathcal{M}$.

Using $\mathbf{v}_i = \text{col}_{k \in \mathcal{T}}(v_{i,k})$ such that $\mathbf{v} = \text{col}_{i \in \mathcal{M}}(\mathbf{v}_i)$, we decompose the proposed formulation in (7) using the following finite horizon scenario program for each subsystem $i \in \mathcal{M}$:

$$\min_{\mathbf{v}_i \in \mathbb{R}^{Tm_i}} J_i(\mathbf{x}_i, \mathbf{u}_i) \quad (10a)$$

$$\text{s.t.} \quad x_{i,k+1}^{(i)} = A_{ii}(\delta_k^{(i)})x_{i,k}^{(i)} + B_{ii}(\delta_k^{(i)})u_{i,k}^{(i)} + q_{i,k}^{(i)}, \quad x_{i,k}^{(i)} = x_{i,0}, \quad (10b)$$

$$x_{i,k+\ell}^{(i)} \in \mathcal{X}_i, \ell \in \mathbb{N}_+, \forall \mathbf{w}_i^{(i)} \in \mathcal{S}_{\mathbf{w}_i}, \forall \delta^{(i)} \in \mathcal{S}_\delta \quad (10c)$$

$$u_{i,k}^{(i)} = K_i x_{i,k}^{(i)} + v_{i,k} \in \mathcal{U}_i, \quad \forall k \in \mathcal{T}, \quad (10d)$$

where $\mathbf{w}_i = \text{col}_{k \in \mathcal{T}}(w_{i,k}) \in \mathcal{W}_i$ such that $\mathcal{W} = \prod_{i \in \mathcal{M}} \mathcal{W}_i$. $\mathcal{S}_{\mathbf{w}_i} := \{\mathbf{w}_i^{(1)}, \dots, \mathbf{w}_i^{(N_{s_i})}\} \in \mathcal{W}_i^{N_{s_i}}$ denotes a set of given finite samples (scenarios) of disturbance in each subsystem $i \in \mathcal{M}$, such that $\mathcal{S}_{\mathbf{w}} = \prod_{i \in \mathcal{M}} \mathcal{S}_{\mathbf{w}_i}$. Note that we use a parenthesis to refer to each scenario of the random variables, e.g. (i) , whereas without the parenthesis refers to each subsystem $i \in \mathcal{M}$.

Remark 7. The proposed constraint (10c) represents an approximation of the following chance constraint on the state of each subsystem $i \in \mathcal{M}$:

$$\mathbb{P}[x_{i,k+\ell} \in \mathcal{X}_i, \ell \in \mathbb{N}_+] \geq 1 - \varepsilon_i, \quad (11)$$

where $\varepsilon_i \in (0, 1)$ is the admissible state constraint violation parameter of each subsystem (9). One can also consider $\alpha_i = 1 - \varepsilon_i$ as the desired level of state feasibility parameter of each subsystem (9).

In the following proposition, we provide a connection between the proposed optimization problem in (10) and the optimization problem in (7).

Proposition 1. Given Assumption 2, the proposed optimization problem in (10) is an exact decomposition of the optimization problem in (7).

Proof. Following the proposed structure of decomposition, any optimizer of each subsystem \mathbf{v}_i^* yields a feasible pair of the state and control input variables of its subsystem $\{\mathbf{x}_i^*, \mathbf{u}_i^*\} \in X_i \times U_i$ such that $X_i = \prod_{k \in \mathcal{T}} \mathcal{X}_i$, and $U_i = \prod_{k \in \mathcal{T}} \mathcal{U}_i$. Therefore, the collection of the optimizers $\mathbf{v}^* = \text{col}_{i \in \mathcal{M}}(\mathbf{v}_i^*)$ will yield the collection of feasible pairs of the state and control input variables of their subsystem:

$$\{\mathbf{x}^* = \text{col}_{i \in \mathcal{M}}(\mathbf{x}_i^*), \mathbf{u}^* = \text{col}_{i \in \mathcal{M}}(\mathbf{u}_i^*)\} \in \left\{ \mathcal{X} = \prod_{i \in \mathcal{M}} X_i \right\} \times \left\{ \mathcal{U} = \prod_{i \in \mathcal{M}} U_i \right\}, \quad (12)$$

which eventually yields a feasible point for the optimization problem in (7). It is straightforward to use the above relation and to show that any optimizer of the optimization problem in (7) \mathbf{v}^* also yields a feasible solution for the proposed optimization problem in (10). We then have to show that both optimization problems will have the same performance index in terms of their objective function values. Due to the proposed decomposition technique, it is easy to see that the objective function in (7) can be formulated as additive components such that each component represents the objective function of each subsystem $i \in \mathcal{M}$, and thus:

$$J(\mathbf{x}^*, \mathbf{u}^*) = \sum_{i \in \mathcal{M}} J_i(\mathbf{x}_i^*, \mathbf{u}_i^*).$$

The proof is completed. \square

The following theorem can be considered as the main result of this section to quantify the robustness of the solutions obtained by (10).

Theorem 2. *Let $\varepsilon_i, \beta_i \in (0, 1)$ for all $i \in \mathcal{M}$ be chosen such that $\varepsilon = \sum_{i \in \mathcal{M}} \varepsilon_i \in (0, 1)$ and $\beta = \sum_{i \in \mathcal{M}} \beta_i \in (0, 1)$. If $\mathbf{v}^* = \text{col}_{i \in \mathcal{M}}(\mathbf{v}_i^*)$, the collection of the optimizer of the problem (10) for all subsystem $i \in \mathcal{M}$, is applied to the discrete time dynamical system (1) for a finite horizon of length T , then, with at least confidence $1 - \beta$, the original chance constraint (5c) is satisfied for all $k \in \mathcal{T}$.*

Proof. The proof is a direct result of combining Proposition 1 with Theorem 1, and uses arguments similar to the ones adopted in [33, Theorem 1]. \square

The interpretation of Theorem 2 is as follows. In the proposed distributed scenario program (10), each subsystem $i \in \mathcal{M}$ can have a desired level of constraint violation ε_i and a desired level of confidence level β_i . To keep the robustness level of collection of the solutions in a probabilistic sense (5c) for the discrete-time dynamical system (1), these choices have to follow a certain design rule, e.g. $\varepsilon = \sum_{i \in \mathcal{M}} \varepsilon_i \in (0, 1)$ and $\beta = \sum_{i \in \mathcal{M}} \beta_i \in (0, 1)$. This yields a fixed ε, β for the large-scale system (1) and the individual ε_i, β_i for each subsystem $i \in \mathcal{M}$.

Remark 8. *An important key feature of the proposed distributed scenario program in (10) compared to the optimization problem in (7) is as follows. Using the proposed distributed framework, we decompose a large-scale scenario program (7) with N_s number of scenarios into $M = |\mathcal{M}|$ small-scale scenario programs (10) with N_{s_i} number of scenarios. This yields a significant reduction in the computation time complexity of scenario programs similar to (7) by using the proposed distributed scenario program (10).*

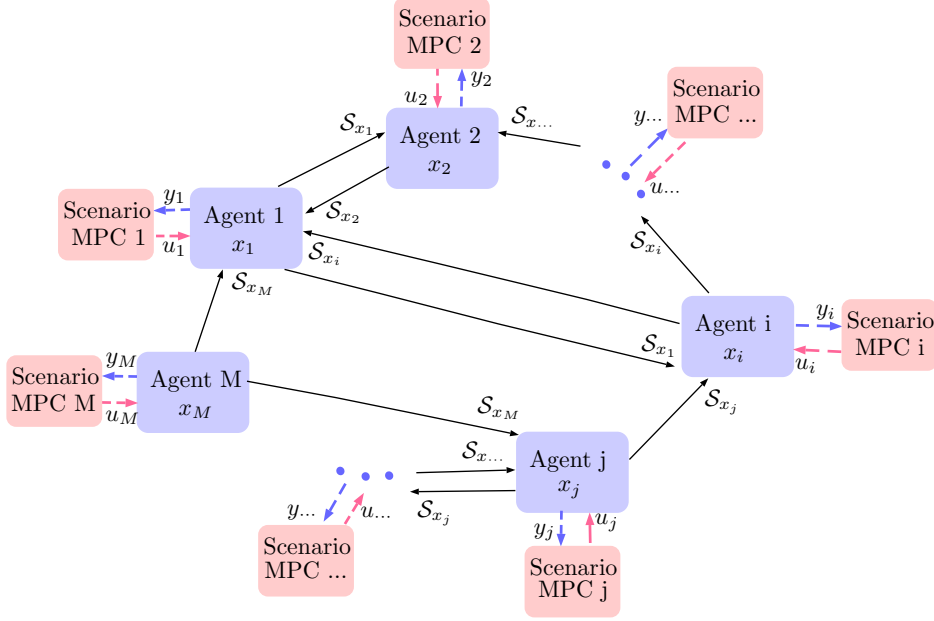


FIGURE 2. Distributed scenario MPC which corresponds to the problem (10).

Using the subsystem dynamics in (9), one can substitute $q_{i,k}^{(i)}$ in the proposed scenario optimization problem (10) with the following relation:

$$q_{i,k}^{(i)} = \sum_{j \in \mathcal{N}_i} A_{ij}(\delta_k^{(i)}) x_{j,k}^{(i)} + C_i(\delta_k^{(i)}) w_{i,k}^{(i)} := f(\delta_k^{(i)}, w_{i,k}^{(i)}, x_{j,k}^{(i)}). \quad (13)$$

$\delta_k^{(i)}$ and $w_{i,k}^{(i)}$ are the local scenario of random variables that are available in each subsystem by definition $w_i^{(i)} \in \mathcal{S}_{w_i}$ and $\delta^{(i)} \in \mathcal{S}_{\delta}$. Hence, the only information that the subsystem $i \in \mathcal{M}$ needs is a N_{s_i} number of samples of the state variable $\mathbf{x}_j^{(i)} = \text{col}_{k \in \mathcal{T}}(x_{j,k}^{(i)}) \in X_j$ from all its neighboring subsystems $j \in \mathcal{N}_i$ at each sampling time $k \in \mathcal{T}$.

Remark 9. It is important to note that even though the proposed distributed scenario program in (10) yields a reduction of computation time complexity (see Remark 8), it however requires more communication between each subsystem, since at each sampling time $k \in \mathcal{T}$ all neighboring agents $j \in \mathcal{N}_i$ of agent i should send a set of scenarios of the state variable $\mathcal{S}_{x_j} := \{\mathbf{x}_j^{(1)}, \dots, \mathbf{x}_j^{(N_{s_i})}\} \in X_j^{N_{s_i}}$ to the agent $i \in \mathcal{M}$.

To summarize this section, we present a network of agents that are dynamically coupled with their own local scenario MPC in Figure 2. In the next section, we will propose a novel inter-agent information exchange scheme to provide a more flexible together with less conservatism framework to exchange information between agents.

4. INTER-AGENT INFORMATION EXCHANGE SCHEME

This section first describes the information exchange between agents and then proposes a set-based information exchange scheme which will be called as a *soft communication protocol* later

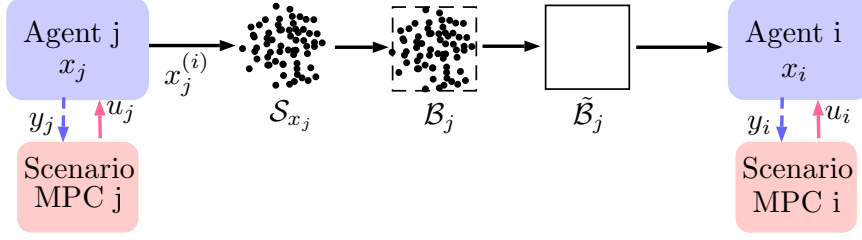


FIGURE 3. Pictorial representation of the proposed inter-agent soft communication scheme. \mathcal{S}_{x_j} is the set of \tilde{N}_{s_i} scenarios of $\mathbf{x}_j^{(i)}$, \mathcal{B}_j is the parametrized set used in the optimization problem (14), and $\tilde{\mathcal{B}}_j$ is the solution of the optimization problem (14).

in this section. We finally provide the theoretical results to quantify robustness of the proposed information exchange scheme between agents.

When the proposed distributed framework (10) is applied to the large-scale scenario program (7), all neighboring agents $j \in \mathcal{N}_i$ of agent $i \in \mathcal{M}$ should send a set of scenarios of the state variable $\mathcal{S}_{x_j} := \{\mathbf{x}_j^{(1)}, \dots, \mathbf{x}_j^{(N_{s_i})}\}$ to the agent i at each sampling time $k \in \mathcal{T}$. It is of interest to address the issue of how an agent $j \in \mathcal{N}_i$ is going to send the contents of \mathcal{S}_{x_j} to the agent $i \in \mathcal{M}$.

We propose the following two schemes:

- a) Following our proposed setup in (10) to achieve a probabilistic guarantee for the obtained solution, agent $i \in \mathcal{M}$ requests from its neighboring agents to send the complete set of data \mathcal{S}_{x_j} , element by element. It is important to mention that the number of required samples N_{s_i} , is chosen according to Theorem 2 in order to have a given probabilistic guarantee for the optimizer \mathbf{v}_i . We refer to this scheme as a *hard communication protocol* between agents. Its advantage is that it is simple and transmits exactly the contents of \mathcal{S}_{x_j} , but due to possibly high values of N_{s_i} it may turn out to be too costly in terms of required communication bandwidth.
- b) To address this shortcoming, we propose another scheme, where instead agent $j \in \mathcal{N}_i$ sends a suitable parametrization of a set that contains all the possible values of its privatized data with a desired level of probability (*the level of reliability*) $\tilde{\alpha}_j$. By considering a simple family of sets, for instance boxes in \mathbb{R}^{m_j} , communication cost can so be kept down at reasonable levels. We refer to this scheme as a *soft communication protocol* between agents (see Fig. 3).

We now describe the soft communication protocol in more details. The neighboring agent $j \in \mathcal{N}_i$ has to first generate \tilde{N}_{s_i} samples of \mathbf{x}_j in order to build the set \mathcal{S}_{x_j} . It is important to notice that in the soft communication protocol the number \tilde{N}_{s_i} of samples generated by agent j may be different than the one needed by agent i , which is N_{s_i} , as will be remarked later. Let us then introduce $\mathcal{B}_j \subset \mathbb{R}^{m_j}$ as a bounded set containing all the elements of \mathcal{S}_{x_j} . We assume for simplicity that \mathcal{B}_j is an axis-aligned hyper-rectangular set. This is not a restrictive assumption and any convex set could have been chosen instead as described in [33]. We can define $\mathcal{B}_j := [-\mathbf{b}_j, \mathbf{b}_j]$ as an interval, where the vector $\mathbf{b}_j \in \mathbb{R}^{m_j}$ defines the hyper-rectangle bounds.

Consider now the following optimization problem that aims to determine the set \mathcal{B}_j with minimal volume:

$$\begin{cases} \min_{\mathbf{b}_j \in \mathbb{R}^{m_j}} & \|\mathbf{b}_j\|_1 \\ \text{s.t.} & \mathbf{x}_j^{(l)} \in [-\mathbf{b}_j, \mathbf{b}_j], \forall \mathbf{x}_j^{(l)} \in \mathcal{S}_{\mathbf{x}_j}, \\ & l = 1, \dots, \tilde{N}_{s_i} \end{cases} \quad (14)$$

where \tilde{N}_{s_i} is a number of samples $\mathbf{x}_j \in \mathcal{S}_{\mathbf{x}_j}$ that neighboring agent j has to take in account in order to determine \mathcal{B}_j . If we denote by $\tilde{\mathcal{B}}_j = [-\tilde{\mathbf{b}}_j, \tilde{\mathbf{b}}_j]$, the optimal solution of (14) computed by the neighbor agent j , then for implementing the soft communication protocol, the agent j needs to communicate only the vector $\tilde{\mathbf{b}}_j$ along with the level of reliability $\tilde{\alpha}_j$ to the agent i .

Definition 1. We call a set $\tilde{\mathcal{B}}_j$ is $\tilde{\alpha}_j$ -reliable if

$$\mathbb{P} \left[\mathbf{x}_j \in X_j : \mathbf{x}_j \notin [-\tilde{\mathbf{b}}_j, \tilde{\mathbf{b}}_j] \right] \leq 1 - \tilde{\alpha}_j, \quad (15)$$

and we refer to $\tilde{\alpha}_j$ as the level of reliability of the set $\tilde{\mathcal{B}}_j$.

We now provide the following theorem to determine $\tilde{\alpha}_j$ as the level of reliability of the set $\tilde{\mathcal{B}}_j$.

Theorem 3. Fix $\tilde{\beta}_j \in (0, 1)$ and let

$$\tilde{\alpha}_j = \tilde{N}_{s_i}^{-m_j} \sqrt{\frac{\tilde{\beta}_j}{\binom{\tilde{N}_{s_i}}{m_j}}}. \quad (16)$$

We then have

$$\mathbb{P}^{\tilde{N}_{s_i}} \left\{ \{\mathbf{x}_j^1, \dots, \mathbf{x}_j^{\tilde{N}_{s_i}}\} \in X_j^{\tilde{N}_{s_i}} : \mathbb{P} \left[\mathbf{x}_j \in X_j : \mathbf{x}_j \notin [-\tilde{\mathbf{b}}_j, \tilde{\mathbf{b}}_j] \right] \leq 1 - \tilde{\alpha}_j \right\} \geq 1 - \tilde{\beta}_j. \quad (17)$$

Proof. Equation (17) is a direct result of the scenario approach theory in [23], if $\tilde{\beta}_j$ is chosen such that

$$\binom{\tilde{N}_{s_i}}{m_j} \tilde{\alpha}_j^{\tilde{N}_{s_i} - m_j} \leq \tilde{\beta}_j. \quad (18)$$

Considering the worst-case equality in the above relation and some algebraic manipulations, one can obtain the above assertion. \square

Theorem 3 implies that given an hypothetical new sample $\mathbf{x}_j \in X_j$, agent $j \in \mathcal{N}_i$ have a confidence of at least $1 - \tilde{\beta}_j$ that the probability of $\mathbf{x}_j \in \tilde{\mathcal{B}}_j = [-\tilde{\mathbf{b}}_j, \tilde{\mathbf{b}}_j]$ is at least $\tilde{\alpha}_j$. Therefore, one can rely on $\tilde{\mathcal{B}}_j$ up to $\tilde{\alpha}_j$ probability.

Remark 10. The number of samples \tilde{N}_{s_j} in the proposed formulation (14) is a design parameter chosen by the neighboring agent $j \in \mathcal{N}_i$. We however remark that one can also set a given $\tilde{\alpha}_j$ as the desired level of reliability and obtain from (16) the required number of samples \tilde{N}_{s_i} .

Remark 11. It is worth to mention that in the proposed optimization problem (14), the performance index is considered to be the minimal volume of the parametrized set \mathcal{B}_j . The reason of such a choice is to determine a less conservative (minimal volume) set that is $\tilde{\alpha}_j$ -reliable. We however note that one can also consider different performance indices, which might not yield the minimal volume, but

it should be taken into account that such a solution will be more conservative. Moreover, the choice of different types (hyper-rectangular, ellipsoidal, and etc.) of the parametrized set \mathcal{B}_j in (14) may also introduce some level of conservativeness.

Remark 12. After receiving the parametrization of $\tilde{\mathcal{B}}_j$ and the level of reliability $\tilde{\alpha}_j$, agent i can then obtain the samples needed for computing $q_{i,k}^{(i)}$ by either locally generating N_{s_i} samples, drawing them uniformly from $\tilde{\mathcal{B}}_j$, or taking the worst-case of $\tilde{\mathcal{B}}_j$. It is important to notice that in this way, we decoupled the sample generation of agent $j \in \mathcal{N}_i$ from the agent $i \in \mathcal{M}$.

When an agent $i \in \mathcal{M}$ and its neighbor $j \in \mathcal{N}_i$ adopt the soft communication scheme, there is an important effect on the probabilistic feasibility of agent i , following Remark 7. Such a scheme introduces some level of stochasticity on the probabilistic feasibility of agent i , due to the fact that the neighboring information is only *probabilistically reliable*. This will affect the local probabilistic robustness guarantee of feasibility as it was discussed in Theorem 2 and consequently in Theorem 1. To accommodate the level of reliability of neighboring information, we need to marginalize a joint cumulative distribution function probability (cdf) of \mathbf{x}_i and the generic sample $\mathbf{x}_j \in X_j$ appearing in Theorem 3. We thus have the following theorem, which can be regarded as the main theoretical result of this section.

Theorem 4. Given $\tilde{\alpha}_j \in (0, 1)$ and a fixed $\alpha_i \in (0, 1)$, the state trajectory of a generic agent $i \in \mathcal{M}$ is probabilistically $\bar{\alpha}_i$ -feasible for all $\mathbf{w}_i \in \mathcal{W}_i$, $\boldsymbol{\delta} \in \Delta$, i.e.,

$$\mathbb{P}[x_{i,k+\ell} \in \mathcal{X}_i, \ell \in \mathbb{N}_+] \geq \bar{\alpha}_i, \quad (19)$$

where $\bar{\alpha}_i = 1 - \frac{1-\alpha_i}{\tilde{\alpha}_i}$ such that $\tilde{\alpha}_i = \prod_{j \in \mathcal{N}_i} (\tilde{\alpha}_j)$.

Proof. The proof consists of two important steps. We first provide an analytical expression for the robustness of the solution in agent i by taking into account the effect of just one neighboring agent $j \in \mathcal{N}_i$, and then extend the obtained results for the case when the agent i interacts with more neighboring agents, e.g. for all $j \in \mathcal{N}_i$.

Following Remark 7, we have the following updated situation:

$$\alpha_i \leq \mathbb{P}[\mathbf{x}_i \in X_i, \mathbf{x}_j \in \tilde{\mathcal{B}}_j],$$

which is a joint probability of $\mathbf{x}_i \in X_i$ and $\mathbf{x}_j \in \tilde{\mathcal{B}}_j$. Such a joint probability can be equivalently written as a joint cumulative distribution function (CDF):

$$\begin{aligned} \alpha &\leq \mathbb{P}[\mathbf{x}_i \in X_i, \mathbf{x}_j \in \tilde{\mathcal{B}}_j] \\ &= \int_{X_i} \int_{\tilde{\mathcal{B}}_j} p(\mathbf{x}_i, \mathbf{x}_j) d\mathbf{x}_i d\mathbf{x}_j \\ &= F_{\mathbf{x}_i, \mathbf{x}_j}(X_i, \tilde{\mathcal{B}}_j), \end{aligned} \quad (20)$$

where $p(\mathbf{x}_i, \mathbf{x}_j)$ is the joint probability density function (PDF) of \mathbf{x}_i and \mathbf{x}_j . Our goal is to calculate:

$$\begin{aligned} \mathbb{P}[\mathbf{x}_i \in X_i] &= \int_{X_i} p(\mathbf{x}_i) d\mathbf{x}_i \\ &= F_{\mathbf{x}_i}(X_i), \end{aligned}$$

where $p(\mathbf{x}_i)$ is the PDF of \mathbf{x}_i . To Transform the joint CDF into the marginal CDF of \mathbf{x}_i , one can take the limit of the joint CDF as $\tilde{\mathcal{B}}_j$ approaches \mathbb{R}^{m_j} :

$$\begin{aligned}
\mathbb{P}[\mathbf{x}_i \in X_i] &= F_{\mathbf{x}_i}(X_i) \\
&= \lim_{\tilde{\mathcal{B}}_j \rightarrow \mathbb{R}^{m_j}} F_{\mathbf{x}_i, \mathbf{x}_j}(X_i, \tilde{\mathcal{B}}_j) \\
&= \lim_{\tilde{\mathcal{B}}_j \rightarrow \mathbb{R}^{m_j}} F_{\mathbf{x}_i | \mathbf{x}_j}(X_i | \tilde{\mathcal{B}}_j) F_{\mathbf{x}_j}(\tilde{\mathcal{B}}_j) \\
&= F_{\mathbf{x}_i}(X_i) \lim_{\tilde{\mathcal{B}}_j \rightarrow \mathbb{R}^{m_j}} F_{\mathbf{x}_j}(\tilde{\mathcal{B}}_j),
\end{aligned} \tag{21}$$

where the last equality is due to the fact that \mathbf{x}_i and \mathbf{x}_j are conditionally independent.

To determine $\lim_{\tilde{\mathcal{B}}_j \rightarrow \mathbb{R}^{m_j}} F_{\mathbf{x}_j}(\tilde{\mathcal{B}}_j)$, one can calculate:

$$\begin{aligned}
\lim_{\tilde{\mathcal{B}}_j \rightarrow \mathbb{R}^{m_j}} F_{\mathbf{x}_j}(\tilde{\mathcal{B}}_j) &= \int_{\mathbb{R}^{m_j}} p(\mathbf{x}_j) d\mathbf{x}_j \\
&= \int_{\mathbb{R}^{m_j} \setminus \tilde{\mathcal{B}}_j} p(\mathbf{x}_j) d\mathbf{x}_j + \int_{\tilde{\mathcal{B}}_j} p(\mathbf{x}_j) d\mathbf{x}_j \\
&= \mathbb{P}[\mathbf{x}_j \notin \tilde{\mathcal{B}}_j] + \mathbb{P}[\mathbf{x}_j \in \tilde{\mathcal{B}}_j] \\
&= (1 - \tilde{\alpha}_j) + \tilde{\alpha}_j \\
&= 1,
\end{aligned} \tag{22}$$

where $p(\mathbf{x}_j)$ is the PDF of \mathbf{x}_j , and the last equality is a direct result of Theorem 3. We now put all the steps together as follows:

$$\begin{aligned}
\alpha_i &\leq \mathbb{P}[\mathbf{x}_i \in X_i, \mathbf{x}_j \in \tilde{\mathcal{B}}_j] \\
&= F_{\mathbf{x}_i, \mathbf{x}_j}(X_i, \tilde{\mathcal{B}}_j) \\
&\leq F_{\mathbf{x}_i}(X_i) \lim_{\tilde{\mathcal{B}}_j \rightarrow \mathbb{R}^{m_j}} F_{\mathbf{x}_j}(\tilde{\mathcal{B}}_j) \\
&= \mathbb{P}[\mathbf{x}_i \in X_i] \left(\int_{\mathbb{R}^{m_j} \setminus \tilde{\mathcal{B}}_j} p(\mathbf{x}_j) d\mathbf{x}_j + \int_{\tilde{\mathcal{B}}_j} p(\mathbf{x}_j) d\mathbf{x}_j \right) \\
&\leq \int_{\mathbb{R}^{m_j} \setminus \tilde{\mathcal{B}}_j} p(\mathbf{x}_j) d\mathbf{x}_j + \mathbb{P}[\mathbf{x}_i \in X_i] \int_{\tilde{\mathcal{B}}_j} p(\mathbf{x}_j) d\mathbf{x}_j \\
&= (1 - \tilde{\alpha}_j) + \tilde{\alpha}_j \mathbb{P}[\mathbf{x}_i \in X_i],
\end{aligned}$$

where the first inequality and equality is due to (20), the second inequality is due to (21), the second and last equality is due to (22), and the last inequality is considering the worst-case situation, e.g. $\mathbb{P}[\mathbf{x}_i \in X_i | \mathbf{x}_j \notin \tilde{\mathcal{B}}_j] = 1$.

By rearranging the last equation in above result:

$$\frac{\alpha_i - (1 - \tilde{\alpha}_j)}{\tilde{\alpha}_j} = 1 - \frac{1 - \alpha_i}{\tilde{\alpha}_j} = \bar{\alpha}_i \leq \mathbb{P}[\mathbf{x}_i \in X_i]. \tag{23}$$

This completes the proof of first part. We now need to show the effect of having more than one neighboring agent. To this end, the most straightforward step, in order to extend the current results, is to use the fact that all neighboring agents are independent from each other. We therefore can apply the previous results for a new situation where the agent i with the probabilistic level of feasibility $\bar{\alpha}_i$ have another neighboring agent $\nu \in \mathcal{N}_i$ with $\tilde{\alpha}_\nu$ the level of reliability of $\tilde{\mathcal{B}}_\nu$. By using Equation (23), we have the following relations for $j, \nu \in \mathcal{N}_i$

$$1 - \frac{1 - \bar{\alpha}_i}{\tilde{\alpha}_\nu} = 1 - \frac{1 - \left(1 - \frac{1 - \alpha_i}{\tilde{\alpha}_j}\right)}{\tilde{\alpha}_\nu} = 1 - \frac{1 - \alpha_i}{\tilde{\alpha}_j \tilde{\alpha}_\nu} \leq \mathbb{P}[\mathbf{x}_i \in X_i] . \quad (24)$$

By continuing the similar arguments for all neighboring agents, one can obtain $\bar{\alpha}_i = 1 - \frac{1 - \alpha_i}{\tilde{\alpha}_i} \leq \mathbb{P}[\mathbf{x}_i \in X_i]$ such that $\tilde{\alpha}_i = \tilde{\alpha}_1 \cdots \tilde{\alpha}_j \tilde{\alpha}_\nu \cdots \tilde{\alpha}_{|\mathcal{N}_i|} = \prod_{j \in \mathcal{N}_i} (\tilde{\alpha}_j)$. The proof is completed. \square

Remark 13. *Following the statement of Theorem 4, it is straightforward to observe that if for all $j \in \mathcal{N}_i$, $\tilde{\alpha}_j \rightarrow 1$ then $\bar{\alpha}_i \rightarrow \alpha_i$. This means that if the level of reliability of the neighboring information is one, $\mathbb{P}[\mathbf{x}_j \in \tilde{\mathcal{B}}_j : \forall j \in \mathcal{N}_i] = 1$, then, the state feasibility of agent i will have the same probabilistic level of robustness as the hard communication scheme, $\mathbb{P}[\mathbf{x}_i \in X_i] \geq \alpha_i$.*

Remark 14. *Following the results in Theorem 4 and taking into account the statement of Theorem 2, the proposed soft communication scheme introduces some level of stochasticity on the feasibility of the large-scale system as in (5c). In particular, $\varepsilon_i \in (0, 1)$ the level of constraint violation in each agent $i \in \mathcal{M}$ will increase, since it is proportional with the inverse of $\prod_{j \in \mathcal{N}_i} (\tilde{\alpha}_j) \in (0, 1)$, and therefore, $\varepsilon = \sum_{i \in \mathcal{M}} \varepsilon_i \in (0, 1)$ will also increase.*

We summarize this section by mentioning that Figure 3 depicts a conceptual representation of the proposed soft communication scheme between two neighborhood agents. In the next section, we provide an operational framework that uses our developments in preceding sections in a more applied framework namely plug-and-play (PnP) operations.

5. PLUG-AND-PLAY OPERATIONAL FRAMEWORK

This section presents a detailed steps of the PnP framework of our proposed approach. It is important to note that the PnP operations are implemented in an offline mode, similar to [13]. We first provide an algorithm to design the distributed scenario MPC for a large-scale dynamical system (1). We then develop two algorithms for the plug-in and plug-out operations of each agent $i \in \mathcal{M}$, respectively.

Algorithm 1 summarizes our proposed approach to distributed scenario MPC. It is assumed here that agents communicate with each other by using our proposed scheme in Section 4, namely the soft inter-agent communication scheme. We however note that in case of the hard communication scheme, each agent needs to generate N_{s_i} samples and send exactly all the samples to all the neighboring agent $j \in \mathcal{N}_i$. In another words, the following changes have to be made in Algorithm 1. \tilde{N}_{s_i} will be substituted by N_{s_i} in Step 7 and Step 8 will be removed. Steps 9 and 10 will send and receive the exactly N_{s_i} samples, respectively, and Step 11 will be also removed.

Algorithm 1 Distributed Scenario MPC

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- 1: **Decompose** the large-scale dynamical system (1) into M agents as the proposed form in (9)
 - 2: **Determine** the index set of neighboring agents \mathcal{N}_i for each agent $i \in \mathcal{M}$
 - 3: **For** each agent $i \in \mathcal{M}$ **do**
 - 4: **fix** $x_{i,k} = x_{i,0} \in \mathcal{X}_i$, $\varepsilon_i \in (0, 1)$, and $\beta_i \in (0, 1)$ such that

$$\varepsilon = \sum_{i \in \mathcal{M}} \varepsilon_i \in (0, 1), \quad \beta = \sum_{i \in \mathcal{M}} \beta_i \in (0, 1) \quad (25)$$
 - 5: **determine** $N_{\bar{s}_i} \in (0, +\infty)$ to approximate the objective function as in (6), and N_{s_i} following Theorem 2 to approximate the chance constraint (11) in an equivalent sense
 - 6: **generate** $N_{\bar{s}_i}$, N_{s_i} scenarios of \mathbf{w}_i , $\boldsymbol{\delta}$ to determine the sets of $\bar{\mathcal{S}}_{\mathbf{w}_i}$, $\bar{\mathcal{S}}_{\boldsymbol{\delta}}$ and $\mathcal{S}_{\mathbf{w}_i}$, $\mathcal{S}_{\boldsymbol{\delta}}$
 - 7: **generate** \tilde{N}_{s_i} scenarios of \mathbf{x}_i using the dynamical system of agent i in form of (9)
 - 8: **determine** set $\tilde{\mathcal{B}}_i$ by solving the optimization problem in (14)
 - 9: **send** the set $\tilde{\mathcal{B}}_i$ to all neighboring agents $j \in \mathcal{N}_i$
 - 10: **receive** the set $\tilde{\mathcal{B}}_j$ from all neighboring agents $j \in \mathcal{N}_i$
 - 11: **generate** the samples of \mathbf{x}_j by drawing them uniformly from $\tilde{\mathcal{B}}_j$, $\forall j \in \mathcal{N}_i$
 - 12: **solve** the proposed optimization problem in (7) and determine a feasible solution \mathbf{v}_i^*
 - 13: **apply** the first input of solution $u_{i,k} = K_i x_{i,k} + v_{i,k}^*$ into the uncertain subsystem (9)
 - 14: **measure** the state and substitute it as the initial state of the next step $x_{i,0}$
 - 15: $k \leftarrow k + 1$.
 - 16: **goto** Step (6)
 - 17: **End for**
-

Remark 15. In Algorithm 1 it is assumed that the feedback control gain matrices K_i for all agent $i \in \mathcal{M}$ are given, following our discussion in (4). In addition, the coupling terms $A_{ij}(\delta)$ are assumed to be known between each agent $i \in \mathcal{M}$ and its neighboring agents $j \in \mathcal{N}_i$.

Algorithm 2 and Algorithm 3 are presented to summarize the steps that are needed for plug-in and plug-out operations, respectively.

Algorithm 2 Plug-in Operation

-
- 1: **Add** the number of new subsystems into the previous number of subsystems, e.g. one additional agent $M \leftarrow M + 1$ such that $|\mathcal{M}| = M + 1$
 - 2: **Update** the index set of neighboring agents \mathcal{N}_i for each agent $i \in \mathcal{M}$
 - 3: **Goto** Step (3) of Algorithm 1
-

Remark 16. When an agent is plugged-in or plugged-out as in Algorithm 2 and Algorithm 3, respectively, one can redesign K_i to may improve the local control performance of each agent $i \in \mathcal{M}$.

Remark 17. In a plugged-in or plugged-out operation as in Algorithm 2 and Algorithm 3, respectively, all agents $i \in \mathcal{M}$ have to update their ε_i with β_i to respect the condition in Theorem 2 as it is shown in (25) to achieve the desired level of constraint feasibility for the large-scale system (1).

Algorithm 3 Plug-out Operation

- 1: **Remove** the number of new subsystems from the previous number of subsystems, e.g. one exclusion agent $M \leftarrow M - 1$ such that $|\mathcal{M}| = M - 1$
 - 2: **Update** the index set of neighboring agents \mathcal{N}_i for each agent $i \in \mathcal{M}$
 - 3: **Goto** Step (3) of Algorithm 1
-

6. NUMERICAL STUDY

This section presents a simulation study to illustrate the functionality of our proposed framework. In order to numerically investigate our proposed approach, we simulate the thermal energy demands of three buildings modeled using realistic parameters and the actual registered weather data in the city center of Utrecht, the Netherlands, where these buildings are located and they had been equipped with aquifer thermal energy storage (ATES) systems [33].

6.1. Simulation Setup

A less well-known sustainable energy storage technology is ATES which is used to store large quantities of thermal energy in aquifers enabling the reduction of energy usage and CO2 emissions of the heating and cooling networks in buildings [34]. An ATES system consists of two wells (warm and cold water wells) and it is considered as a heat source or sink, or as a storage for thermal energy that operates in a seasonal mode.

Following the developed model in [33], each ATES system of each building is modeled using first-order difference equations, leading a large-scale network of interconnected agents (buildings) that are dynamically coupled via the state variables of ATES systems, which are the volume of water and the thermal energy content of each well. In order to prevent overlap of nearby systems, there are constraints on a growing thermal radius, $r_{i,k}^h$ [m], $r_{i,k}^c$ [m], of each well of an ATES system with its neighboring agents, e.g. $r_{i,k}^h + r_{j,k}^c \leq d_{ij}$ for each agent $i \in \mathcal{M}$ and its neighboring agent $j \in \mathcal{N}_i$. In [33, Corollary 1], it was shown that such a constraint on the growing thermal radius can be replaced with a constraint on the dynamically coupled state variables (volume of water) of the ATES systems. We refer interested readers to [33] for the detailed explanations on the case study.

We here take some steps forward to model a more realistic ATES system by introducing additive disturbance and multiplicative uncertainty sources into the deterministic dynamical model of [33, 34]. It has been reported in [35, 36] that the ambient temperature of water in aquifer is changing by the long life-time usage of ATES systems. We capture such a behavior by introducing an additive unknown (disturbance) source of energy which yields a time correlation function via the initial value of energy content variable of an ATES system. In addition to this, the system parameters of an ATES system are a function of the stored water temperature in each well, e.g. see [33, Figure 2]. We therefore introduce a small level of perturbation as an uncertainty source in the parameters of the ATES system dynamics.

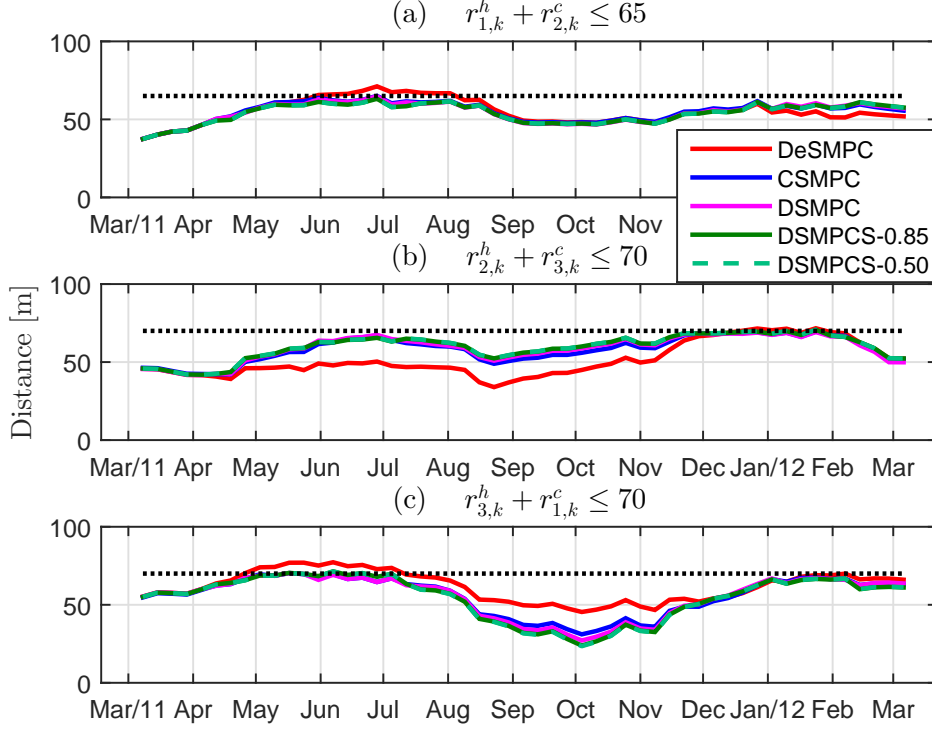


FIGURE 4. A-posteriori feasibility validation of the obtained results. The "red", "blue", "magenta", "solid green", and "dashed green" lines are related to the obtained results via DeSMPC, CSMPC, DSMPC, DSMPCS–0.85, and DSMPCS–0.50, respectively. The "dotted black" lines are related to the upper bound values.

To generate random scenarios from the additive disturbance, we built a discrete normal process such that the nominal scenario is 10% of the amplitude of the energy content in a deterministic ATES system model and varies within 5% of its nominal scenario at each sampling time. As for the multiplicative uncertainty, we generate a random variable from a Gaussian distribution with a mean value 0, variance 0.3 and a maximal magnitude of 0.03 at each sampling time. The simulation environment was MATLAB with YALMIP as the interface [37] and Gurobi as the solver.

6.2. Simulation Results

Simulation results are delivered using a hierarchy control scheme in which the upper layer is used to coordinate between ATES systems to meet the dynamically coupled constraint in a weekly-based time steps with a three months prediction horizon, whereas in the lower layer, we solved a SMPC for individual buildings to keep track of the desired temperature of each building using the proposed formulation in [33, Problem 11]. We here focus on the results obtained in the upper layer, since the lower layer problems are decoupled and the results were similar to the one in [33, Figure 6].

To illustrate the advantages of our proposed framework, we simulate four problem formulations, namely a centralized SMPC (CSMPC) using the formulation in (7), a distributed SMPC (DSMPC) via the distributed scenario program in (10), and DSMPC with the proposed soft communication scheme with 0.85–reliability (DSMPCS–0.85) as described in Definition 1 and DSMPCS–0.50, both

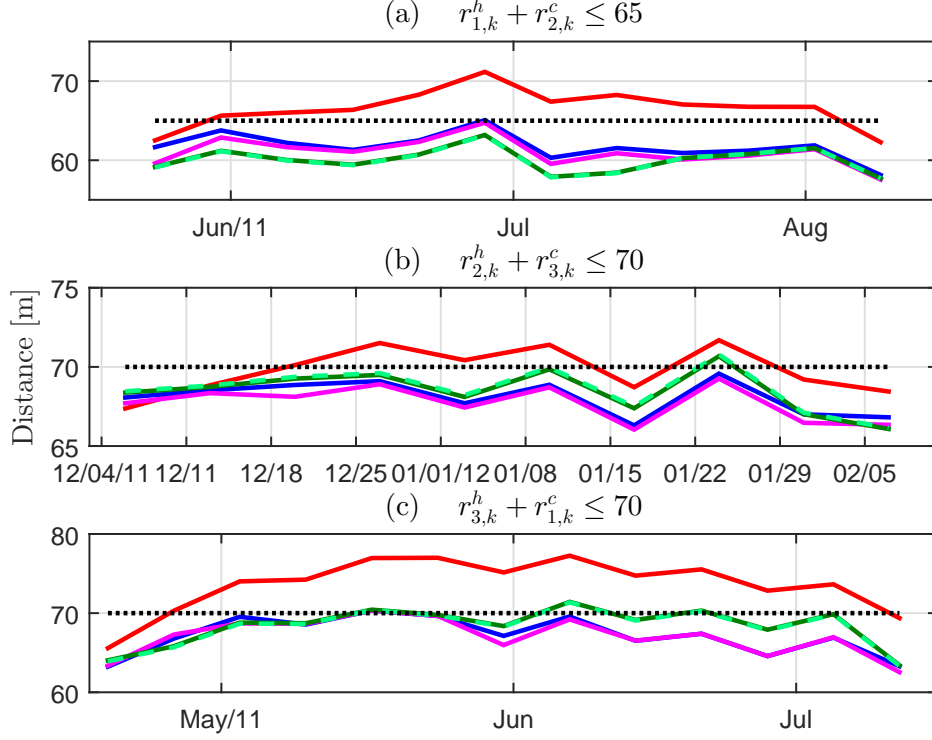


FIGURE 5. Zoom-in of the critical time periods in Fig. 4.

following Algorithm 1. For comparison purposes, we also present the results obtained via decoupled SMPC (DeSMPC), where the coupling constraints are relaxed. The simulation study is carried out for one year using actual weather conditions from March 2011-March 2012 and the proposed formulations in a closed-loop control system framework.

In Fig. 4 we show an a-posteriori feasibility validation of the dynamically coupled constraints between each agent $i = 1, 2, 3$, and neighboring agents, e.g. $r_{1,k}^h + r_{2,k}^c \leq 65$, $r_{2,k}^h + r_{3,k}^c \leq 70$, and $r_{3,k}^h + r_{1,k}^c \leq 70$. The "red" line is related to the results obtained via DeSMPC, the "blue" line shows the results obtained via CSMPC, the "magenta" presents the results obtained by using DSMPC, the "solid green" and "dashed green" lines show the results obtained via DSMPCS-0.85 and DSMPCS-0.50, respectively. The "dotted black" lines are related to the upper bound values of these three coupling constraints. Fig. 5 focuses on the critical time periods in Fig. 4, where both neighboring agents are injecting thermal energies with different pump flow rates. Fig. 5(a) shows the results from mid-May to mid-August 2011, Fig. 5(b) presents the results of December 2011 to February 2012, and Fig. 5(c) depicts the results of mid-April to mid-July 2011.

As a first desired achievement using our proposed frameworks, one can clearly see in Fig. 4 the dynamically coupled states trajectories are feasible in a probabilistic sense, since the agent operations are quite close to the upper bounds in the critical time periods compared to DeSMPC that violates the constraints. Strictly speaking, using our proposed framework one can achieve the maximum usage of the aquifer (subsurface) to store thermal energy without affecting the neighboring thermal storage. This is a direct result of Theorem 1, such that the obtained solutions via our proposed formulations

have to be probabilistically feasible, that can be clearly seen in Fig. 5, since the trajectories are on the upper bounds almost surely.

It is worth to mention that Fig. 5 illustrates our other two main contributions more precisely: 1) the obtained results via CSMPC (blue line) and DSMPC (magenta line) are practically equivalent throughout the simulation; this is due to Proposition 1 and Theorem 2. Actually, the obtained solutions via DSMPC are slightly more conservative compared to the obtained results via CSMPC, and this is a direct consequence of Theorem 2. In fact the level of violation in CSMPC is considered to be $\varepsilon = 0.05$ and leading to $\varepsilon_i = 0.0167$ for all agents due to Theorem 2, which is more restrictive. 2) using the proposed soft communication scheme yields less conservative solutions as explicitly derived in Theorem 4, and can be clearly seen in Fig. 5 with the obtained results via DSMPCS-0.85 (solid green) and DSMPCS-0.50 (dashed green). Following Theorem 4 the new violation level using DSMPCS-0.85 is $\bar{\varepsilon}_i = 0.0231$, and using DSMPCS-0.50 is $\bar{\varepsilon}_i = 0.0668$. It is important to notice that the violation level of global chance constraint will increase to $\bar{\varepsilon} = 0.0702$ and $\bar{\varepsilon} = 0.2004$ using DSMPCS-0.85 and DSMPCS-0.50, respectively, following Remark 14.

7. CONCLUDING REMARKS AND FUTURE WORK

In this paper we presented a rigorous approach to distributed stochastic model predictive control (SMPC) using the scenario-based approximation. We then provided a novel inter-agent soft communication scheme to minimize the amount of information exchange between each subsystem. Using a set-based parametrization technique, we introduced a probabilistically reliable notation and quantified the robustness of the obtained distributed SMPC solution via the integrated soft communication scheme. The probabilistic guarantees of the proposed distributed SMPC framework coincide with its centralized counterpart.

Our current research direction concentrates on extending the proposed framework to cope with the case of a common uncertainty source. It is important to point out that the discussion on scenario optimization was limited to constraint satisfaction probability. An interesting future work is to analyze the performance index by means of optimality of the scenario program solutions in distributed setting by exploiting the results in [38]. Finally, another interesting future research direction is to investigate the possibility of extending our proposed frameworks to the fault detection and isolation problems following the results in [39, 40].

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